JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 22, No. 5, September–October 1999

Multiple-Spacecraft Orbit-Transfer Problem: The No-Booster Case

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Primer vector theory, in combination with a parameter optimization algorithm, is used to compute the optimal transfer of *n* spacecraft from an initial parking orbit to a final operational orbit with the added constraint that all spacecraft are injected from the parking orbit on one upper-stage booster and that all are required to be spaced along the final orbit according to some prescribed, but otherwise arbitrary, spacing constraint between individual spacecraft. A particular case of the problem, known as the no-booster case, is examined, in which it is assumed that all spacecraft are required to perform all of the necessary maneuvers without the aid of a booster and the spacecraft are simply constrained to start at the same location on the initial orbit. The solution is formulated for a general force field, and examples are given for a three-spacecraft constellation transfer in the restricted three-body problem force-field model.

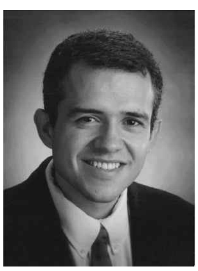
Introduction

SEVERAL researchers have addressed the optimization of transfer trajectories for multiple spacecraft. Some of these studies have examined the cooperative rendezvous problem in which both spacecraft take an active role in the rendezvous with the possibility of further reducing propellant consumption over the usual active–passive case. Mirfakhraie and Conway[1] examined time-fixed impulsive rendezvous between active spacecraft in a central force field. Differential cost gradients were developed for different transfer cases in which the spacecraft performed one or more impulsive maneuvers to perform the rendezvous. Propellant constraints were included, and the problem was solved with an unconstrained parameter optimization algorithm. Coverstone-Carroll and Prussing[2] addressed a similar problem of cooperative rendezvous for spacecraft equipped with variable-thrust (power-limited) propulsion systems. Both the Clohessy-Wiltshire linear force-field model and the inverse-square central force-field model were used. The optimal control problem was solved by means of a direct numerical approach that required discretization of the states and the controls.

For a general cooperative rendezvous problem, several spacecraft begin on different orbits and end up moving along the same final orbit that can be different from any of the original starting orbits. The multiple-spacecraft orbit-transfer problem can be seen as the reverse of the cooperative rendezvous problem as all of the spacecraft begin at the same location on the same orbit and are required to move to different points along some final orbit or a set of final orbits.

The problem treated here requires placing an arbitrary number of spacecraft at different locations along the same final orbit. The impulsive thrust approximation is used to model the finite burn maneuvers. The final orbit is arbitrary, but, without loss of generality, assume it is periodic with period T_p. Assume further that all of the spacecraft are injected onto the transfer orbit by an upper-stage booster. Clearly, for the inverse-square force field, a two-impulse transfer between circular orbits would yield the Hohman transfer as the optimal two-impulse transfer for one spacecraft; the transfer of n spacecraft between these orbits subject to the final-point spacing constraint is not entirely trivial especially when the orbits are general and the force-field model is more complicated. Figure 1 shows a schematic of the general single-booster multiple-spacecraft orbit-insertion problem.

The necessary conditions for an optimal transfer of one spacecraft between two arbitrary orbits that can be assumed to be a parking



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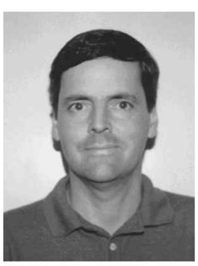

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Received 18 December 1997; revision received 23 December 1998; accepted for publication 26 February 1999.

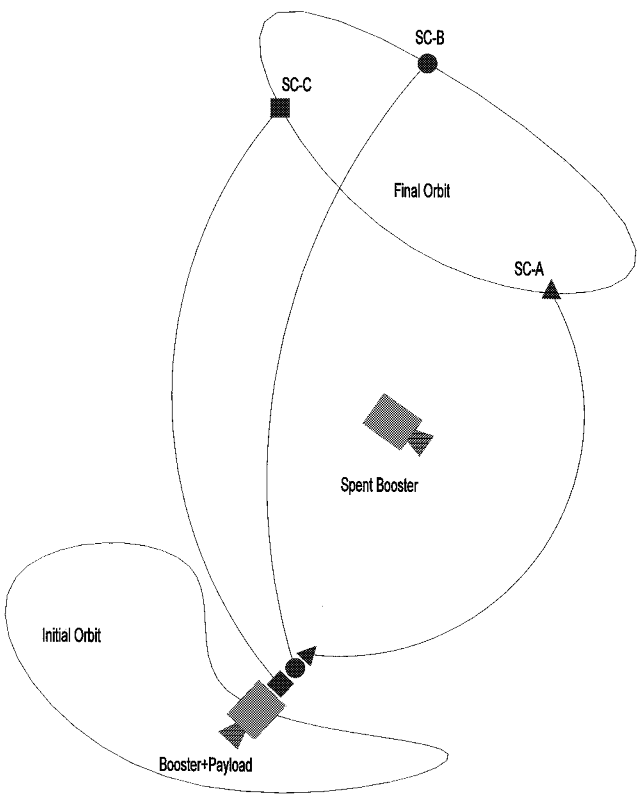

Fig. 1 Single-booster multiple-spacecraft orbit-transfer problem. SC, spacecraft.

orbit and a final operational orbit has been well documented in the literature, and their extension to the restricted three-body problem (RTBP) force-field model are documented in a recent dissertation.[3] The transfer of multiple spacecraft to the final orbit for which the initial departure maneuver in the parking orbit must be provided by an upper stage is an extension of the single-spacecraft transfer problem with the added constraints that result from the nature of the departure maneuver (booster plus spacecraft stack) and the final orbit-insertion maneuver(s) required of all spacecraft for satisfying the spacing constraint. This approach treats the problem as a constrained orbit-transfer problem for several spacecraft. An alternative approach to the problem is to model it as a pseudorendezvous problem in which the spacecraft are required to rendezvous with fictitious spacecraft moving along the final orbit already in their respective relative separations. The first approach is the one treated in this study, although it would be interesting to compare and examine the final results with the formulation and the associated algorithms for the pseudorendezvous case.

The performance index can take one of several forms. For example, for a given booster, assuming that all n spacecraft have the same engine characteristics, the problem is to maximize the sum of the final masses of each spacecraft on orbit, or, equivalently, to minimize the sum of all of the Δv maneuvers required by all of the spacecraft if each spacecraft is identical. Identical spacecraft implies that each is equipped with the same propulsion system and each has the same initial mass. If the propellant cost of the booster is included, the performance index to minimize can be given by

$$J = kJ_b + \sum_{i=1}^{n} J_i \tag{1}$$

where J_b is the propellant cost for the booster, k ($k \geq 0$) is a weighting factor that relates the comparative engine efficiencies between that of the individual spacecraft and the booster, and J_i is the propellant cost for each individual spacecraft. In this analysis, $k = 0$, implying that the cost of the booster maneuver is unimportant, even though the parameters describing this maneuver can still serve as independent parameters in reducing the value of the remaining part of the cost function.

The current problem can also be stated and solved as a minimax optimization problem by minimization of the sum of the impulsive maneuvers made by the spacecraft with the highest maneuver requirement, i.e.,

$$\min\left[\max\left(\Delta v_T^1, \Delta v_T^2, \ldots, \Delta v_T^n\right)\right]$$

where Δv_T^i is the sum of all of the impulsive velocity magnitudes made by spacecraft i. Note that this is a composite function of smooth functions that in general is nondifferentiable, although it can be transformed into a smooth optimization problem. Although these cost functions are distinct and specific to certain problems, the procedure to construct the gradient of the cost function with respect to the independent parameters and optimize a nominal solution is the same.

In this initial study, the optimization problem solved is simply to minimize the sum of the total Δv made by all of the spacecraft:

$$J = \sum_{i=1}^{n} \Delta v_T^i$$

If the target orbit is periodic with time period T_p and it is required to place n spacecraft equally spaced in time along this orbit, then one of the final position constraints takes the form

$$\Delta\tau_{1\to 2} = T_p/n$$

where $\Delta\tau_{1\to 2}$ is the distance in time between spacecrafts 1 and 2. If all spacecraft perform at most two maneuvers, the parameter or search vector in the most general case is of the form

$$z^T = \left(\Delta \boldsymbol{v}_0^B \ \ \tau_0^B \ \ \tau_m^1 \ \ \tau_m^2 \ \ \cdots \ \ \tau_m^n \ \ T^1 \ \ T^2 \ \ \cdots \ \ T^n \ \ \tau_f^1 \ \ \tau_f^2 \ \ \cdots \ \ \tau_f^n\right)$$

where $\Delta \boldsymbol{v}_0^B$ is the booster impulsive maneuver, τ_0^B represents the coast time for the booster on the initial orbit, τ_m^i represents the coast time for each spacecraft along the intermediate transfer orbit, T^i is the total time taken by each spacecraft between the initial and the final maneuvers, i.e., the time of flight for each spacecraft between the transfer orbit and the final orbit, and τ_f^i are the arrival locations for each spacecraft on the final orbit, which are not necessarily independent. The arrival locations τ_f^i are expressed in terms of a normalized coasting time from some reference point on the final orbit. The characteristic time length used to normalize the coasting time is taken to be the final orbit's time period, if it is periodic for example. It is possible that the individual spacecraft may need to perform more than two maneuvers to reduce the cost of the transfer further. In this case, if p represents the required number of interior impulses required for each spacecraft for accomplishing the transfer optimally, then $\boldsymbol{t}_p^i$ would be a vector composed of the times at which these impulses occur for the ith spacecraft and $\boldsymbol{r}_p^i$ are the locations in Cartesian coordinates of these intermediate impulses.

As stated, the problem is divided into the following three cases:

1) Case 1, no booster: This case considers the transfer of n spacecraft between two arbitrary orbits without the aid of an upper-stage booster. All spacecraft perform the required number of impulses to complete the transfer subject to the final spacing constraint. This is a trivial case that is used as a basis for the other two cases. In this case, all of the spacecraft move from the initial orbit O^0 to the final orbit O^f, and the booster does nothing.

2) Case 2, weak booster: In this case, all spacecraft are injected onto an intermediate transfer orbit O^m with the aid of an upper-stage booster. After injection, all spacecraft perform the number of impulses required for moving from the transfer orbit to the final mission orbit. The primary assumption for this case is that the intermediate trajectory does not intersect the final orbit, and thus all spacecraft are required to perform at least two maneuvers to complete the transfer. This case results when a propellant constraint is imposed on the booster, thus bounding the maximum injection Δv capability for a given payload. As a consequence, the transfer orbit may not reach the final orbit and hence the designation as the weak-booster case. This case is used when the upper-stage booster parameters are fixed and it is required to determine the optimal conditions for the transfer

of the n spacecraft by using their own dedicated propulsion systems to move from O^m to O^f.

3) Case 3, strong booster: This case is similar to case II without the booster propellant constraint, therefore making it possible for the booster transfer orbit to intersect the final orbit at least once, so that at least one spacecraft can complete the transfer with a single impulse. If the injection orbit intersects the final orbit more than once, as is the case if it is periodic, for example, then it may even be possible for all spacecraft to perform a single impulse to complete the transfer. This case is used when it is desired to size the upper-stage booster for a given payload of n spacecraft. In this case, the booster and the first spacecraft move from O^0 to O^f by means of O^m and the remaining spacecraft move from O^m to O^f.

Case 1 is treated here. A cost is defined, and the differential of the cost is derived in the form of a cost gradient such that the coefficients of the independent variations are the elements of the gradient vector that can be used in an unconstrained optimization algorithm. Further examination of each coefficient provides information about the conditions that must be satisfied on a stationary solution to the transfer problem. The value and the sign of these coefficients also provide information on how a nominal solution can be improved at least to first order by a slight adjustment of the independent variables in the appropriate direction. The method described here can be useful for the optimization of future satellite constellation transfer problems that are currently being planned. Although the examples assume a RTBP force-field model, these can be extended and applied to any force field for which the solution to this problem is required.

Restricted Three-Body Force Model

A complete description of the circular RTBP can be found in Ref. 4. In this study, the RTBP is used to approximate the motion of a spacecraft in the sun–Earth–moon system. The smaller mass is the combined Earth and moon masses and is referred to as the Earth. The Earth and the sun are assumed to be in circular orbits about their common barycenter. The vector equation of motion centered at the Earth is of the form

$$\ddot{\boldsymbol{r}} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) \tag{2}$$

where the vector quantities $\boldsymbol{g}$ and $\boldsymbol{h}$ depend on only either position or velocity:

$$\boldsymbol{g}(\boldsymbol{r}) \equiv \boldsymbol{f}_e - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{3}$$

$$\boldsymbol{h}(\boldsymbol{v}) \equiv -2\boldsymbol{\omega} \times \boldsymbol{v} \tag{4}$$

where $\boldsymbol{f}_e$ is the total external force per unit mass,

$$\boldsymbol{f}_e = -\left[(1-\mu)/r_1^3\right]\boldsymbol{r}_1 - (\mu/r^3)\boldsymbol{r}$$

$\boldsymbol{\omega}$ is the angular velocity of the rotating frame, μ is the RTBP mass ratio parameter, and $\boldsymbol{r}_1$ and $\boldsymbol{r}$ are the position vectors from the larger mass (sun) and the smaller mass (Earth plus moon) to the spacecraft, respectively. These quantities are illustrated in Fig. 2.

When the state vector is defined as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix}$$

the first-order form of the equations of motion becomes

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{pmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) \end{bmatrix} \tag{5}$$

Linearization along a nominal trajectory leads to the linear variational equations

$$\delta\dot{\boldsymbol{x}} = F(t)\delta\boldsymbol{x} \tag{6}$$

where $F(t)$ is the time-varying state propagation matrix given by the Jacobian matrix of the vector field:

$$F(t) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ G_{3\times3} & H_{3\times3} \end{bmatrix}$$

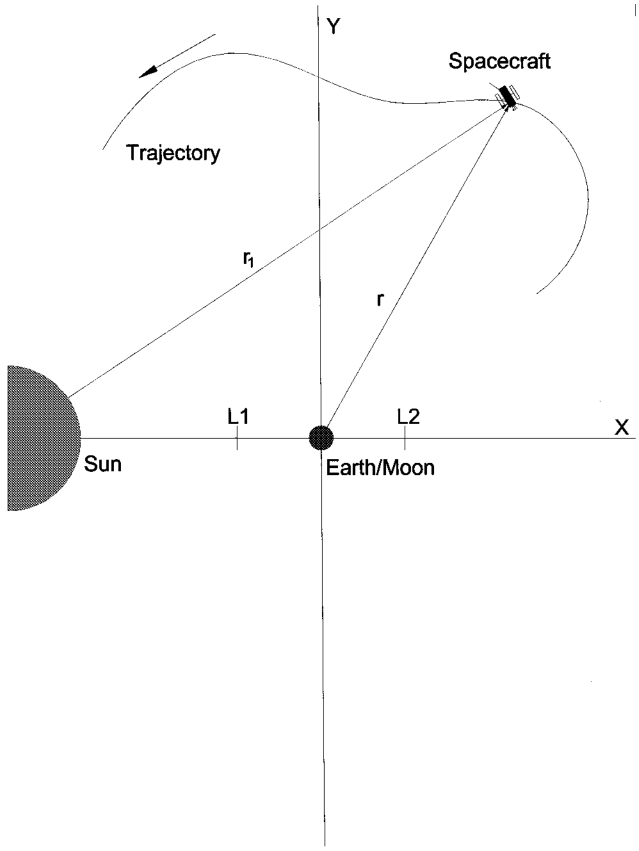

Fig. 2 Earth-centered RTBP coordinate system.

where $I_{3\times3}$ is a 3×3 identity matrix, G is a symmetric gravity-gradient matrix, and H is a general velocity-gradient matrix. The submatrix G takes the form

$$G = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}} = \frac{(1-\mu)}{r_1^5}\left(3\boldsymbol{r}_1\boldsymbol{r}_1^T - r_1^2 I_{3\times3}\right) + \frac{\mu}{r^5}\left(3\boldsymbol{r}\boldsymbol{r}^T - r^2 I_{3\times3}\right) - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times I_{3\times3}) \tag{7}$$

The G matrix is referred to as the symmetric gravity-gradient matrix for the RTBP; except for the last two terms, this matrix is equivalent to the gravity-gradient matrix for the inverse-square (two-body) force-field model. The submatrix H is given by

$$H = -2\boldsymbol{\omega} \times I_{3\times3} \tag{8}$$

Note that $G^T = G$ but that $H^T = -H$. The linear variational equations can than be expressed as

$$\begin{pmatrix} \delta\dot{\boldsymbol{r}} \\ \delta\dot{\boldsymbol{v}} \end{pmatrix} = \begin{pmatrix} \delta\dot{\boldsymbol{r}} \\ \delta\ddot{\boldsymbol{r}} \end{pmatrix} = \begin{bmatrix} 0 & I \\ G & H \end{bmatrix} \begin{pmatrix} \delta\boldsymbol{r} \\ \delta\boldsymbol{v} \end{pmatrix}$$

so that the second time derivative of the position deviation is

$$\delta\ddot{\boldsymbol{r}} = G\delta\boldsymbol{r} + H\delta\boldsymbol{v} \tag{9}$$

The solution to this linear system takes the form[5]

$$\delta\boldsymbol{x}(t) = \Phi(t, t_i)\delta\boldsymbol{x}(t_i)$$

where $\Phi(t, t_i)$ is the state-transition matrix that satisfies the matrix differential equation

$$\dot{\Phi}(t, t_i) = F(t)\Phi(t, t_i)$$

where initial condition $\Phi(t_i, t_i) = I_{6\times6}$ and $I_{6\times6}$ is the identity matrix of dimension 6. This system is integrated along a nominal trajectory, and $\Phi(t, t_i)$ is used to map small perturbations $\delta\boldsymbol{x}(t_i)$ forward or backward in time.

Optimization of an Impulsive Thrust Trajectory

The application of optimal control theory to the optimization of impulsive trajectories leads to primer vector theory.[6] The transfer problem is cast as a parameter optimization problem with analytically constructed gradients. The application to impulsive transfers in the central force-field problem has been well documented in the literature.[7–10] The major results are summarized here.

The cost function for the impulsive trajectory problem is equivalent to minimizing the sum of all interior and terminal impulses:

$$\min J = \Delta v_{\text{total}} = \sum_{i=1}^{n} |\Delta \boldsymbol{v}_i| \tag{10}$$

where n is the number of impulses. Along a coasting (null-thrust) arc between impulses, the equations of motion are

$$\begin{pmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) \end{bmatrix} \tag{11}$$

The linearized solution for small deviations about a nominal path with initial conditions $\boldsymbol{r}_0^T\ \boldsymbol{v}_0^T$ is given by

$$\begin{bmatrix} \delta\boldsymbol{r}(t) \\ \delta\boldsymbol{v}(t) \end{bmatrix} = \Phi(t, t_0) \begin{bmatrix} \delta\boldsymbol{r}(t_0) \\ \delta\boldsymbol{v}(t_0) \end{bmatrix} \tag{12}$$

where $\Phi(t, t_0)$ is the state-transition matrix solution of the variational equations associated with Eq. (11) and serves as a linear mapping between t_0 and some other earlier or later time t. The primer and primer rate vectors, $\boldsymbol{p}(t)$ and $\dot{\boldsymbol{p}}(t)$, respectively, satisfy the same equations as the state variations, namely,

$$\begin{pmatrix} \dot{\boldsymbol{p}} \\ \ddot{\boldsymbol{p}} \end{pmatrix} = \begin{bmatrix} 0 & I \\ G & H \end{bmatrix} \begin{pmatrix} \boldsymbol{p} \\ \dot{\boldsymbol{p}} \end{pmatrix} \tag{13}$$

so that the solution for $\boldsymbol{p}$ and $\dot{\boldsymbol{p}}$ is

$$\begin{bmatrix} \boldsymbol{p}(t) \\ \dot{\boldsymbol{p}}(t) \end{bmatrix} = \Phi(t, t_0) \begin{bmatrix} \boldsymbol{p}(t_0) \\ \dot{\boldsymbol{p}}(t_0) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} \begin{bmatrix} \boldsymbol{p}(t_0) \\ \dot{\boldsymbol{p}}(t_0) \end{bmatrix} \tag{14}$$

where $\Phi(t, t_0)$ has been partitioned into 3×3 submatrices Φ_{ij}. Assume a two-impulse segment of a more general multiimpulse transfer and let the first impulse occur at $t = t_0$ and the second impulse occur at $t = t_f$. To satisfy Lawden's necessary conditions, the primer vector at each terminal time is set to a unit vector in the direction of the impulse:

$$\boldsymbol{p}(t_0) = \Delta\boldsymbol{v}_0/\Delta v_0, \qquad \boldsymbol{p}(t_f) = \Delta\boldsymbol{v}_f/\Delta v_f$$

and as a consequence the primer magnitude is unity at these impulse times, $\boldsymbol{p}(t_0) = \boldsymbol{p}(t_f) = 1$. This leads to a linear two-point boundary-value problem that is readily solved. When Eq. (14) is used, the primer vector derivative is given at t_0 and t_f as

$$\dot{\boldsymbol{p}}_0 = \Phi_{12}^{-1}(\boldsymbol{p}_f - \Phi_{11}\boldsymbol{p}_0)$$

$$\dot{\boldsymbol{p}}_f = \left(\Phi_{21} - \Phi_{22}\Phi_{12}^{-1}\Phi_{11}\right)\boldsymbol{p}_0 + \left(\Phi_{22}\Phi_{12}^{-1}\right)\boldsymbol{p}_f$$

where all of the submatrices of the state-transition matrix are evaluated from t_0 to t_f. In the second of these equations it is assumed that Φ_{12}^{-1} exists. On a coasting arc, the Hamiltonian of the augmented state–costate system is[10]

$$H = \dot{\boldsymbol{p}}^T\boldsymbol{v} - \boldsymbol{p}^T\boldsymbol{g} \tag{15}$$

Further, the expression

$$\lambda^T\delta\boldsymbol{x} = \lambda_r^T\delta\boldsymbol{r} + \lambda_v^T\delta\boldsymbol{v} = (\dot{\boldsymbol{p}}^T + \boldsymbol{p}^T H)\delta\boldsymbol{r} - \boldsymbol{p}^T\delta\boldsymbol{v} \equiv \text{const} \tag{16}$$

is referred to as the adjoint equation. The expressions for the Hamiltonian and the adjoint equation between impulses are used to derive the gradients of the cost with respect to the independent variations. The values of these gradients on a stationary solution are zero, and setting these analytical expressions to zero can be interpreted as necessary conditions for an optimal impulsive transfer.

The general orbit-transfer problem is defined as the transfer of a spacecraft from some given initial orbit to a final orbit at which the departure and the arrival locations on the orbits are considered free and are specified by a position referencing variable τ. The only requirement for this to be valid is to assume a force-field vector $\boldsymbol{f}$ that is time independent, as is the case for the RTBP force-field model in rotating coordinates. The total transfer time between the first and the last impulse is given by $t_f - t_0$ and is also considered a free parameter. Let τ_0 be the departure location from the initial orbit O^0 and τ_f be the arrival location on the final orbit O^f. The optimization problem is stated as follows: For given initial and terminal orbits O^0 and O^f, determine the initial and the final times t_0 and t_f, respectively, the initial and the final departure and arrival locations τ_0 and τ_f, respectively, and, if required, the time and the location of the n intermediate impulse(s) t_{m_i} and $\boldsymbol{r}_{m_i}$ $(i = 1, \ldots, n)$, such that the sum of the magnitudes of all of the impulses is a minimum.

If T_0 and T_f are defined to be some characteristic time-length unit on the initial and the final orbits, then τ is the normalized state referencing variable on either the initial or final orbits. If T_1 is the total transfer time between Δv_0 and Δv_m and T_2 is the transfer time between Δv_m and Δv_f then the differential of the cost $\mathrm{d}J$ is given by

$$\mathrm{d}J = T_0 H_0^- \,\mathrm{d}\tau_0 + H_{0m}\,\mathrm{d}T_1 + \left(\dot{\boldsymbol{p}}_m^{+T} - \dot{\boldsymbol{p}}_m^{-T}\right)\mathrm{d}\boldsymbol{r}_m + H_{mf}\,\mathrm{d}T_2 - T_f H_f^+\,\mathrm{d}\tau_f \tag{17}$$

where H_{0m} and H_{mf} are the values of the Hamiltonian on the segments $[t_0, t_m^-)$ and $(t_m^+, t_f]$, respectively, and the superscripts $-$ and $+$ refer to the quantity's value before and after the impulse, respectively, at the times labeled by the subscripts 0, m, and f. If the terminal orbits are periodic, then T_0 and T_f can be taken to be the time periods for these orbits. If intermediate impulses are excluded, then the differential of the cost is

$$\mathrm{d}J = T_0 H_0^-\,\mathrm{d}\tau_0 + H_{0f}\,\mathrm{d}T - T_f H_f^+\,\mathrm{d}\tau_f \tag{18}$$

where T is the total transfer between the impulses Δv_0 and Δv_f.

The optimal parameter set for a minimum of J is found by means of an unconstrained optimization algorithm because the boundary conditions are inherently satisfied at each value of the parameter set. This is because all of the required Lambert solutions are obtained at each value of the parameter set.

Case 1: The No-Booster Case

Consider the problem of transferring n spacecraft from an intermediate orbit O^0 to a final orbit denoted as O^f. At t_0 all spacecraft are at the same location on O^0. All spacecraft are identical, and each is equipped with its own propulsion system. It is required to place this group of spacecraft on the final orbit with the constraint that each be separated in time by an equal amount $\Delta\tau$.

With the first spacecraft as the reference spacecraft and assuming that its arrival point τ_f^1 is unconstrained, the arrival points of the remaining $n-1$ spacecraft, τ^j $(j = 2, \ldots, n)$, are required to satisfy the following constraint equations:

$$\tau_f^j = \tau_f^1 + (T_{p0}/T_{pf})\left(\tau_0^j - \tau_0^1\right) + (1/T_{pf})(T^j - T^1) + (j-1)\Delta\tau \qquad j = 2, \ldots, n \tag{19}$$

where T_{p0} and T_{pf} are characteristic time-length units that in this case are the orbit periods of the intermediate and the final orbits, respectively, τ_0^i $(i = 1, \ldots, n)$ are the departure points on O^0, T^j $(j = 2, \ldots, n)$ are the transfer times between the departure and the arrival impulses, and $\Delta\tau$ is the specified (normalized) spacing between each spacecraft on the final orbit. For n spacecraft there are $n - 1$ spacing constraints. The objective is to maximize the sum of the final spacecraft masses on orbit. Assuming impulsive maneuvers for all spacecraft, the cost is given by

$$J = \min \sum_{i=1}^{n} \frac{\sum_{k=1}^{l^i} \Delta v_k^i}{c_i}$$

where Δv_k^i is the magnitude of the kth maneuver for spacecraft i and l^i is the number of impulses made by spacecraft i. If all of the spacecraft have the same engine efficiency $c_i = c$ and each spacecraft is allowed at most two maneuvers, then the cost function simplifies to

$$J = \min \sum_{i=1}^{n} \sum_{k=1}^{2} \Delta v_k^i \tag{20}$$

When the cost for spacecraft i is defined as

$$J^i = \sum_{k=1}^{2} \Delta v_k^i$$

the total cost to minimize for the n spacecraft is

$$J = \sum_{i=1}^{n} J^i$$

For a two-impulse transfer, the differential of the cost function for the optimal orbit transfer for any single spacecraft i is

$$\mathrm{d}J^i = T_{p0} H_0^- \,\mathrm{d}\tau_0 + H \,\mathrm{d}T - T_{pf} H_f^+ \,\mathrm{d}\tau_f \tag{21}$$

where the superscripts $-$ and $+$ on the Hamiltonian refer to the value before and after the impulse, respectively, at the times labeled by the subscripts 0 and f. When this result is generalized to n spacecraft, the differential of J is

$$\mathrm{d}J = T_{p0} \sum_{i=1}^{n} H_0^{-i} \,\mathrm{d}\tau_0^i + \sum_{i=1}^{n} H^i \,\mathrm{d}T^i - T_{pf} \sum_{i=1}^{n} H_f^{+i} \,\mathrm{d}\tau_f^i \tag{22}$$

where the optimization variables are not all independent as some of these are subject to the $n-1$ spacing constraints. The differentials of the dependent parameters $\mathrm{d}\tau_f^j$ $(j = 2, \ldots, n)$ are given by

$$\mathrm{d}\tau_f^j = \mathrm{d}\tau_f^1 + (T_{p0}/T_{pf})\left(\mathrm{d}\tau_0^j - \mathrm{d}\tau_0^1\right) + (1/T_{pf})(\mathrm{d}T^j - \mathrm{d}T^1) \qquad (j = 2, \ldots, n)$$

After these dependent variations are eliminated, the differential of the cost for the no-booster case is

$$\mathrm{d}J = T_{p0}\left(H_0^{-1} + \sum_{i=2}^{n} H_f^{+i}\right)\mathrm{d}\tau_0^1 + T_{p0}\sum_{i=2}^{n}\left(H_0^{-i} - H_f^{+i}\right)\mathrm{d}\tau_0^i + \left(H^1 + \sum_{i=2}^{n} H_f^{+i}\right)\mathrm{d}T^1 + \sum_{i=2}^{n}\left(H^i - H_f^{+i}\right)\mathrm{d}T^i - T_{pf}\left(\sum_{i=1}^{n} H_f^{+i}\right)\mathrm{d}\tau_f^1 \tag{23}$$

The independent optimization variables are grouped into the parameter vector

$$z^T = \left(\tau_0^1 \quad \cdots \quad \tau_0^n, T^1 \quad \cdots \quad T^n, \tau_f^1\right)$$

so that for n spacecraft there are $3n$ parameters with $n-1$ constraints, yielding a total of $2n+1$ independent parameters to define a transfer uniquely. The coefficients of the independent variations can be used in a multidimensional parameter optimization algorithm as the gradient vector to obtain a stationary solution to the problem. The problem is solved as an unconstrained parameter optimization problem as the spacing constraint has been eliminated algebraically and all of the kinematic boundary conditions at each value of the parameter vector are satisfied by means of the associated sequence of generalized Lambert solutions for each spacecraft.

Examination of the gradients on a stationary solution indicate that the following conditions must be satisfied on a locally optimal solution:

$$H_0^{-1} = -\sum_{i=2}^{n} H_f^{+i}, \qquad H^1 = -\sum_{i=2}^{n} H_f^{+i}, \qquad H_f^1 = -\sum_{i=2}^{n} H_f^{+i}$$

$$H^i = H_f^{+i} \qquad (j = 2, \ldots, n)$$

$$H_0^{-i} = H_f^{+i} \qquad (j = 2, \ldots, n)$$

which together imply that

$$H_0^{-i} = H^i = H_f^{+i} \qquad (j = 1, \ldots, n)$$

so that the Hamiltonian evaluated on the trajectories of each individual spacecraft is constant and continuous although not necessarily zero as is the case for a single-spacecraft transfer.

Transfer Example for the No-Booster Case

To illustrate the application of the cost gradient expression given by Eq. (23), this section presents transfer examples between two RTBP trajectories. The numerical optimization algorithm used is the nonlinearly constrained parameter optimization routine VF13 from the Harwell Subroutine Library.[11] The numerical parameters for this RTBP system are given in Table 1.

The example transfers are between a distant direct orbit (DDO) and a distant retrograde orbit (DRO). These orbits are part of a general class of orbits known as distant Earth orbits, which are defined to be periodic or quasi-periodic orbits centered on the Earth–moon system and exist beyond the moon's orbit up to distances of the order of millions of kilometers. These orbits exist in a restricted three-body force model in which the primary bodies are the sun and the Earth–moon system. In the rotating reference frame associated with the RTBP, a DDO is centered on the Earth and a spacecraft on this trajectory orbits in a counterclockwise sense when viewed from a point north of the ecliptic plane. A spacecraft on a DRO orbits in a clockwise sense. These orbits are useful for spacecraft missions that require a large, yet bounded, Earth separation distance.

The parameters for the initial and the final orbits are given in Table 2. The spacecraft (SC) all have an initial mass of 1000 kg with an I_{sp} of 500 s and are labeled as SC-A, SC-B, and SC-C. The spacing requirement on the final orbit is specified to be 10% of the final orbit period $\Delta\tau = 0.1$.

The results and the parameters for the nominal and the optimal transfer are given and compared in Table 3. In this and other tables, the departure and the arrival points are given as fractions of the orbital period of the initial and the final orbits that have been normalized to unity and zero is the epoch at which the initial conditions are specified. The nominal transfer is based on a set of parameter values that yields a feasible but nonoptimal transfer. The optimal solution is shown in Fig. 3. The cost decreased sharply after the first few iterations, and there was no appreciable improvement in

Table 1 Sun–Earth–moon system RTBP parameters (International Astronautical Union, 1976)

Parameter	Symbol	Value	Units
Sun gravitational parameter	Gm_s	$1.32712438 \times 10^{20}$	m^3/s^2
Earth + moon gravitational parameter	Gm_e	$4.03503294 \times 10^{14}$	m^3/s^2
Distance unit (1 astronomical unit)	a	$1.49597870 \times 10^{11}$	m
Mean motion (calculated)	n	$1.99098670 \times 10^{-7}$	rad/s
RTBP mass ratio (calculated)	μ	$3.04042389 \times 10^{-6}$	——

Table 2 Initial and final orbit parameters for multiple-spacecraft transfer examples

Orbit	Description	x_0, km	$\dot{y}_0$, m/s	T_p, days
Initial orbit O^0	DDO	500,000	814.955	47.2921
Final orbit O^f	DRO	5,000,000	−2014.914	345.9780

Table 3 Results for the no-booster transfer case

Spacecraft separation, $\Delta\tau = 0.1$	Symbol	Nominal	Optimal	Units
SC-A departure point	τ_0^A	1.0000	1.1566	Fraction O^0 period
SC-B departure point	τ_0^B	1.0000	1.1316	Fraction O^0 period
SC-C departure point	τ_0^C	1.0000	1.0959	Fraction O^0 period
SC-A flight time	T^A	300.00	301.31	days
SC-B flight time	T^B	300.00	203.44	days
SC-C flight time	T^C	300.00	226.58	days
SC-A arrival point	τ_f^A	0.9000	1.1264	Fraction O^f period
SC-A maneuver magnitude	Δv^A	2261.25	845.75	m/s
SC-B maneuver magnitude	Δv^B	1887.20	801.73	m/s
SC-C maneuver magnitude	Δv^C	1191.03	945.40	m/s
Total of maneuver magnitudes	Δv^T	5338.30	2592.88	m/s
SC-A final mass	m_f^A	630.55	841.57	kg
SC-B final mass	m_f^B	680.53	849.16	kg
SC-C final mass	m_f^C	784.35	824.64	kg

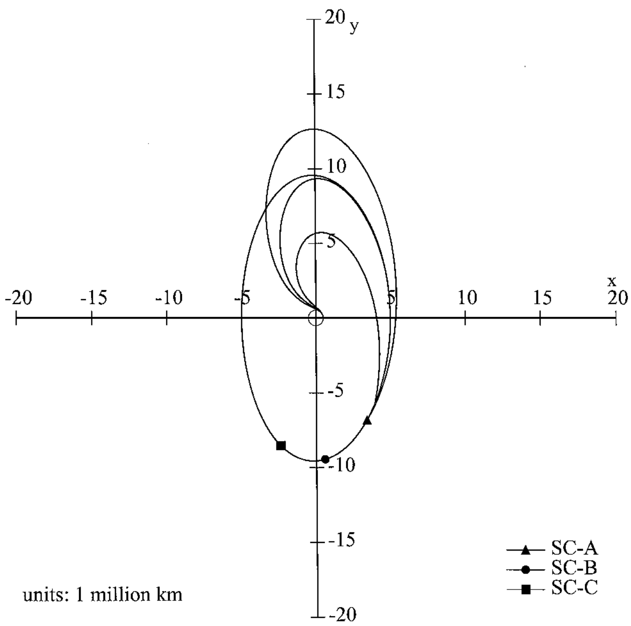

Fig. 3 Optimal transfer for the no-booster case.

the cost after the fifteenth iteration. The key result from this transfer solution is that all of the spacecraft final masses are relatively similar as their corresponding maneuver costs are equally weighted in the optimization process.

In the figures showing the constellation transfer, only the final positions of all of the spacecraft are shown. This point occurs when the last spacecraft makes its final insertion maneuver, and therefore the other spacecraft have already arrived on the final orbit and have coasted to the locations shown. In other words, the arrival points of each spacecraft are not necessarily the same as their locations at the time when the transfer is complete. The final spacecraft locations shown satisfy the spacing constraint, and this is easily discerned from these figures.

Transfers for a Multiple-Spacecraft Interferometer Constellation

This section presents the low Earth orbit (LEO)-DRO transfer of a three-spacecraft constellation required to be spaced out equally in time along a DRO about the Earth–moon system. The solution of this problem as a no-booster case provides a basis for comparison with the solution by use of the assumptions of the weak- and the strong-booster cases.

There are at least a couple of space physics missions that are considering the use of multiple spacecraft to serve as an interferometer for the detection of gravity waves.[12,13] The example transfer presented here is similar to that required by the mission discussed by Hellings et al.[12] except that the final orbit in this case is the same DRO discussed in the preceding example. This example is referred to as a multiple-spacecraft interferometer constellation (MSIC) transfer. The initial spacecraft masses are 1000 kg, and the specific impulse for each is 500 s. The spacing requirement is to space the three spacecraft evenly along the periodic orbit so that $\Delta\tau = \frac{1}{3}$.

Table 4 Initial and final orbit parameters for LEO-DRO transfer example

Orbit	Description	x_0, km	$\dot{y}_0$, m/s	T_p, days
Initial orbit O^0	LEO	6659.053	7782.823	0.0622
Final orbit O^f	DRO	5,000,000	−2014.914	345.978

Table 5 Parameters for LEO-DRO class A1 transfer

Parameter	Symbol	Value	Units
Departure point	τ_0	0.9751	Fraction O^0 period
Transfer time	T	255.27	days
Arrival point	τ_f	1.0469	Fraction O^f period
Initial maneuver magnitude	Δv_0	3250.60	m/s
Final maneuver magnitude	Δv_f	119.32	m/s
Total maneuver magnitude	Δv_T	3369.92	m/s

LEO-DRO Transfer for a Single Spacecraft

The simplest LEO-DRO transfer is constructed from a restricted three-body trajectory that subsequently returns to the Earth if the DRO insertion maneuver is not made. The departure conditions for this transfer orbit are chosen such that it is tangent to the DRO at the point that both orbits cross the x axis. This type of LEO-DRO transfer is referred to as a class A1 transfer.[3] The initial orbit is a LEO that is circular, and the target orbit is a simple periodic DRO about the Earth–moon system. In these examples, motion is confined to the x, y plane, which, for the sun–Earth–moon RTBP model, is the ecliptic plane. This assumption primarily simplifies the presentation of the results even though the formulation discussed in this analysis allows for motion along three axes.

To approximate and simulate the injection conditions from LEO, an approximate launch model used by Pu and Edelbaum[14] is used in this study. A 200-km circular parking orbit about the Earth is approximated by assuming a circular parking orbit about the Earth + moon mass at an altitude at which the inertial circular orbit velocity is equal to the circular orbit velocity of the original 200-km circular orbit about the Earth alone. The initial and the final orbit parameters are given in Table 4.

The optimized Class A1 transfer for the single spacecraft is shown in Fig. 4, and the result summary for this transfer is tabulated in Table 5. Figure 4 also presents the time evolution of the primer vector magnitude. The primer magnitude is unity at all of the impulse times although this may not be clearly seen at t_0 as the primer magnitude time rate is large and negative at t_0 for this particular example.

The results and the parameters for the nominal and the optimal transfers are given in Table 6. The optimal solution is shown in

Table 6 No-booster transfer results

Spacecraft separation, $\Delta\tau = \frac{1}{3}$	Symbol	Nominal	Optimal	Units
SC-A departure point	τ_m^A	1.0000	0.9628	Fraction O^m period
SC-B departure point	τ_m^B	1.5000	1.4973	Fraction O^m period
SC-C departure point	τ_m^C	1.5000	1.3935	Fraction O^m period
SC-A flight time	T^A	250.00	153.17	days
SC-B flight time	T^B	250.00	364.17	days
SC-C flight time	T^C	250.00	187.03	days
SC-A arrival point	τ_f^A	1.0000	0.8512	Fraction O^f period
SC-A maneuver magnitude	Δv^A	3488.84	3450.96	m/s
SC-B maneuver magnitude	Δv^B	4175.41	3400.75	m/s
SC-C maneuver magnitude	Δv^C	4223.39	3764.49	m/s
Total maneuver magnitude	Δv^T	11888.00	10616.17	m/s
SC-A final mass	m_f^A	490.90	494.70	kg
SC-B final mass	m_f^B	426.75	499.79	kg
SC-C final mass	m_f^C	422.60	464.06	kg

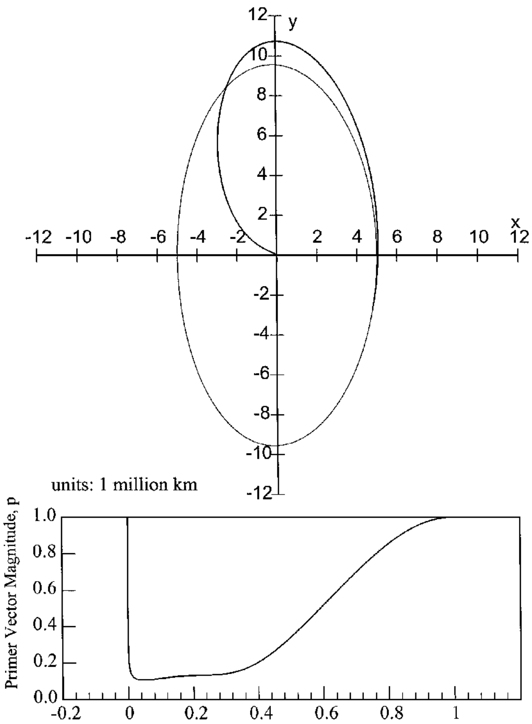

Fig. 4 Minimum Δv_{total} two-impluse transfer for a single spacecraft.

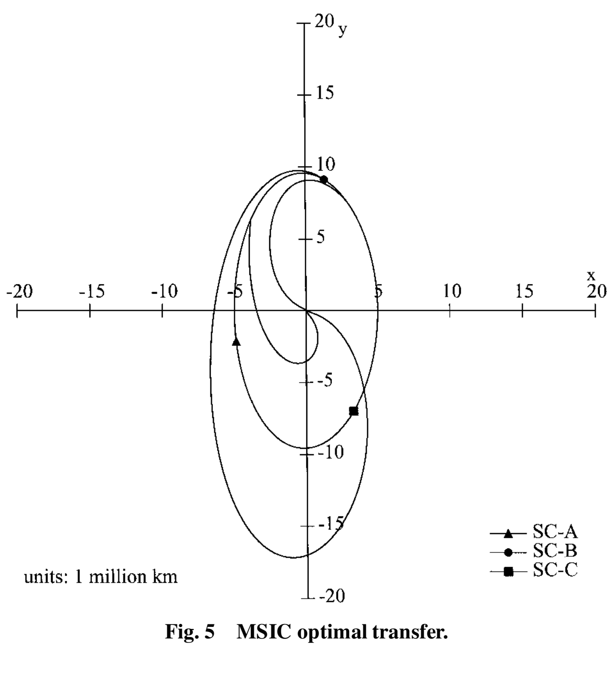

Fig. 5 MSIC optimal transfer.

Fig. 5. One would expect that the impulsive cost for each spacecraft would be greater than the impulsive cost associated with a single spacecraft because of the final spacing constraint. In Table 5, the two-impulse class A1 transfer required a total impulsive cost of 3369.92 m/s. This value should be the lower bound on the impulsive cost associated with each spacecraft for this transfer case. As expected, the spacecraft with the minimum impulsive cost turns out to be spacecraft B with a combined impulsive cost of 3400.75 m/s.

Conclusions

The simplest case of the multiple-spacecrafttransfer problem has been presented. When the cost was taken to be the sum of all of the impulsive maneuvers made by each spacecraft, the differential of the cost with respect to the independent parameters was derived with primer vector theory. This provides analytical cost gradient equations that can be used in an unconstrained parameter optimization algorithm to obtain transfer solutions for n spacecraft equally spaced in time along a final orbit. The formulation restricted the number of impulses for each spacecraft to two; however, it may be possible to lower the maneuver costs by allowing multiple impulses for each transfer trajectory. This would require an additional set of independent search parameters such as positions and times for these intermediate impulses if it is determined that an intermediate impulse can lower the transfer cost.

For the MSIC transfer solved here, the Δv costs for each of the spacecraft are all nearly equal for the optimal transfer; however, an interesting result is that the minimum of these is still greater then the Δv cost of the optimal two-impulse transfer for a single spacecraft. These general results provide the foundation for an extended study that considers the weak-booster and the strong-booster cases.

Acknowledgments

This work is part of C. A. Ocampo's graduate research at the University of Colorado, Boulder, during the period 1992–1996, which was supported by a NASA Graduate Student Researchers Program Fellowship. The support of the fellowship and the individuals associated with it is gratefully acknowledged.

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